PEB
2019

Group A

1. [25 marks: $6 + 7 + 12$]

   Ms. A earns Rs. 25,000 in period 1 and Rs. 15,000 in period 2. Mr. B earns Rs. 15,000 in period 1 and Rs. 30,000 in period 2. They can borrow money at an interest rate of 200%, and can lend money at a rate of 0%. They like both consumption in period 1 (C_1) and consumption in period 2 (C_2), and their preferences are such that their chosen consumption bundles will always lie on their budget lines.

   (a) [6 marks]

   Write down the equations of their budget constraints and draw their budget lines in the same figure by plotting consumption in period 1 (C_1) (in thousand rupees) on x-axis and consumption in period 2 (C_2) (in thousand rupees) on y-axis.

   (b) [7 marks]

   Given the income profile and the market interest rates, Mr. B chooses to borrow Rs. 5,000 in period 1.

   Give an example of a consumption profile (that is, (C_1, C_2)) such that, if Ms. A chooses this profile, we would know for sure that Ms. A and Mr. B have **different preferences** for consumption in period 1 (C_1) and consumption in period 2 (C_2). Give a clear explanation for your answer.

   (c) [12 marks: $6 + 6$]

   Suppose now that Ms. A and Mr. B have the **same preferences** for C_1 and C_2, and, as in part (b), Mr. B borrows Rs. 5,000 in period 1.

i. Suppose that Ms. A chooses to be a lender in period 1. Find out, with a clear explanation, the *maximum* amount that she will lend in period 1 consistent with the fact that they have the same preferences for C_1 and C_2.

ii. Explain clearly whether Mr. B is better off than Ms. A.

2. [25 marks: $6 + 12 + 7$]

Consider an exchange economy with two agents 1 and 2 and two goods X and Y. There is one unit of both goods in the economy. An allocation is a pair $\{(X_1, Y_1), (X_2, Y_2)\}$ where $X_1 + X_2 = 1$, $Y_1 + Y_2 = 1$, and (X_1, Y_1) and (X_2, Y_2) are the consumption bundles of agents 1 and 2 respectively. The utility function for agent 1 is given by $u_1(X_1, Y_1) = X_1 \cdot Y_1$ and that of agent 2 by $u_2(X_2, Y_2) = 2X_2 + Y_2$.

(a) [6 marks]

Describe the set of Pareto-efficient allocations in the economy.

(b) [12 marks: $6 + 6$]

An allocation $\{(X_1, Y_1), (X_2, Y_2)\}$ is *envy-free* if no agent strictly prefers the consumption bundle of the other agent to her own, that is, $u_1(X_1, Y_1) \geq u_1(X_2, Y_2)$ and $u_2(X_2, Y_2) \geq u_2(X_1, Y_1)$.

i. Consider each of the two statements below. Decide whether they are true or false. Justify your answer with a proof or a counter-example as appropriate.

A. All Pareto-efficient allocations are envy-free.

B. All envy-free allocations are Pareto-efficient.

ii. Describe the set of envy-free allocations in the economy.

(c) [7 marks]

Suppose each agent has an endowment of half-unit of each good. Prove **without direct computation** that the competitive equilibrium allocation is both Pareto-efficient and envy-free.

3. [25 marks: $7 + 3 + 3 + 12$]

Two flat-mates, 1 and 2, rent a flat and play their own music on the only CD player owned by the flat-owner. They both like their own music, but dislike the music played by the other person. Given the timing constraints, each one must play her own music when the other person is also present. Let m_i denote the amount of music played by i, and Y_i denote her amount of money holding. Individual i's utility function is

$$u_i(m_1, m_2, Y_i) = 8m_i - 2m_i^2 - \frac{3}{2}m_j^2 + Y_i, \quad i, j = 1, 2, \ i \neq j.$$

(a) [7 marks]

How much music would each individual play? What is the efficient amount of music for each individual? Is the amount of music actually played more or less than the efficient level? Explain the economic intuition for your answer.

(b) [3 marks]

Suppose that individual 2 is considering to gift a headphone to her flat-mate on her birthday. Assume that she does not get any utility from just gift-giving. What is the maximum price she is willing to pay for the headphone?

(c) [3 marks]

Suppose that the price of the headphone is Rs. 11. Does it make sense for the two flat-mates to jointly buy a headphone, sharing the price equally, and making a binding commitment that they would each listen to their own music only via the headphone?

(d) [12 marks]

Now suppose that the CD player is owned by individual 1 so that she can prevent individual 2 from playing any music at all. Suppose individual 1 can offer a *take-it-or-leave-it* contract that looks like the following:

"I shall play music at a level $\bar{m}_1$, and you can play music at the level $\bar{m}_2$ in return for a sum of Rs. T."

In case the offered contract is rejected, individual 1 selects m_1 unilaterally, and individual 2 cannot play any music of her choice. Solve for the optimal levels of $\bar{m}_1$, $\bar{m}_2$ and T. Discuss the economic intuition for your answer.

Group B

1. [25 marks: $2 + 10 + 13$]

   Consider a country where there are only two provinces – A and B. The production function to produce a single output Y is given by $Y = F(N^A + N^B)$ where F is a concave function and N^i represents employees from province i, $i = A, B$. Wages paid to the employees are given by W^i, $i = A, B$. Price of the final good Y is denoted by P. The employers are price takers and take P, W^A and W^B as given.

   (a) [2 marks]

   Write down the expression for an employer's profit as a function of N^A and N^B, $\pi\left(N^A, N^B\right)$.

   (b) [10 marks]

   An employer chooses N^A and N^B to maximize

   $$u\left(N^A, N^B\right) = u\left(\pi\left(N^A, N^B\right), N^A, N^B\right),$$

   where $\frac{\partial u}{\partial \pi} > 0$, $\frac{\partial u}{\partial N^A} > 0$ and $\frac{\partial u}{\partial N^B} < 0$. The last two conditions on $u\left(N^A, N^B\right)$ imply that the employer prefers employees from province A but dislikes employees from province B.

   Write down the first order conditions for the employer's maximization problem assuming an interior solution.

   (c) [13 marks]

   In equilibrium do the employees from different provinces get the same wage? If yes, explain your answer. If not, then determine, with a clear explanation, which employees are paid more and by how much.

2. [25 marks: $4+5+8+8$]

Consider a concave utility function $u(c,l)$ where c represents consumption good and l represents labour supply (working hours, to be precise). While utility increases with the level of consumption good, increasing working hours reduces utility. Wage per hour of labour is given by w, thus working for l hours will ensure wl amount of total wage which is denoted by y, that is, $y = wl$. Given this, the utility function can be written as $u(c, \frac{y}{w})$. The price of the consumption good c is given by p. Also $\overline{L}$ is a fixed number of hours representing total time available to an agent and $\overline{L} - l$ represents leisure. [In all the figures you are asked to draw below, plot y on x-axis and c on y-axis.]

(a) [4 marks]

Derive the slope of an indifference curve for the utility function $u(c, \frac{y}{w})$ on the y-c plane.

(b) [5 marks]

Demonstrate the agent's utility maximizing choice of y and c in a figure by plotting her budget line and indifference curves for the utility function $u(c, \frac{y}{w})$.

(c) [8 marks]

Experiment 1: Suppose there is an increase in w. Demonstrate the agent's new utility maximizing choice of y and c in the same figure as in part (b). [Show clearly how the agent's budget line and/or indifference curves change as a result of the increase in w.] Compare the old and new choices with a brief economic explanation.

(d) [8 marks]

Experiment 2: Suppose, instead of an increase in w, there is a tax imposed on income. That is, the after-tax income of the agent is $(1-\tau)\,y$ where τ is the proportional tax rate. In a new figure demonstrate the agent's new as well as old (as in part (b)) utility maximizing choices of y and c. [Show clearly how the agent's budget line and/or indifference curves change as a result of this proportional tax.] Compare the old and new choices with a brief economic explanation.

3. [25 marks: $4+3+10+4+4$]

Consider an agent who lives for three periods but consumes only in periods two and three where the consumptions are denoted by c_2 and c_3 respectively. Her utility is given by $u\,(c_2, c_3) = \log(c_2) + \beta \log(c_3)$, where $0 < \beta < 1$ is the discount factor reflecting her time preference. She invests an amount e in education in the first period which she borrows from the market at a given interest rate $r > 0$. Her income in the second period is $w \cdot h(e)$ where w is a fixed wage rate per unit of human capital and $h(e)$ is the amount of human capital that results from investment in education (e) in the first period. Assume that $h(e)$ is an increasing and concave function of e. The agent repays her education loan in the second period. She has no income in the third period. But she can save (s) in the second period from her income on which she receives the return $s\,(1+r)$ in the third period to meet her consumption expenditure.

(a) [4 marks]

Write down the agent's period 2 and period 3 budget constraints separately.

(b) [3 marks]

Set up the agent's utility maximization problem by showing her choice variables clearly.

(c) [10 marks]

Write down the first order conditions for the agent's utility maximization problem.

(d) [4 marks]

Derive the ratio of consumptions in period 2 and period 3, $\frac{c_2}{c_3}$, in terms of the parameters of the model.

(e) [4 marks]

Explain how investment in education, e, depends on the preference parameter β.

Group C

1. [25 marks: $5 + 10 + 10$]

   Consider a street represented by the interval $[0, 1]$. Three agents, $\{1, 2, 3\}$, live on this street. Agent $i \in \{1, 2, 3\}$ lives at $x_i \in [0, 1]$, and assume that $x_1 \leq x_2 \leq x_3$. Suppose we locate a hospital at a point $p \in [0, 1]$.

   (a) [5 marks]

   We say p is **square-optimal** if it minimizes $\sum_{i=1}^{3}(x_i - p)^2$. Derive the square optimal value of p.

   (b) [10 marks]

   We say p is **absolute-optimal** if it minimizes $\sum_{i=1}^{3} |x_i - p|$.

   i. Argue that if p is absolute-optimal, then $p \in [x_1, x_3]$.

   ii. Use this to derive an absolute-optimal p.

   (c) [10 marks]

   Now suppose that n agents, $\{1, 2, 3, ..., n\}$, live on this street where $x_1 \leq x_2 \leq x_3 ... \leq x_n$ and n is an odd number. Derive an absolute-optimal p.

2. [25 marks: $7 + 5 + 5 + 8$]

   Two random variables x_1 and x_2 are uniformly drawn from $[0, 1]$. Define the following function:

$$G(p) = p \times \text{Probability}[\ p \geq \max(x_1, x_2)] \qquad \forall\ p \in [0, 1].$$

   (a) [7 marks]

   Derive, with a clear explanation, the expression for $G(p)$.

(b) [5 marks]

Plot $G(p)$.

(c) [5 marks]

Is G convex or concave in p? Give clear explanations for your answer.

(d) [8 marks]

Find $\max_{p\in[0,1]} G(p)$.

3. [25 marks: $7+9+9$]

Let $X \subset \mathbb{R}$ and $f : X \to X$ be a continuous function.

(a) [7 marks]

Suppose $X = [0,1]$. By using the Intermediate Value Theorem, show that there exists $x^* \in X$ such that $f(x^*) = x^*$.

(b) [9 marks]

In each of the cases below, determine whether there exists $x^* \in X$ such that $f(x^*) = x^*$. Justify your claim by either providing a proof or a counter-example.

i. $X = (0,1)$ and f is continuous.

ii. $X = [0,1] \cup [2,3]$ and f is continuous.

iii. $X = [0,1]$ but f is not continuous.

(c) [9 marks]

Let $f_i : [0,1] \to [0,1]$, $i = 1, 2, \ldots, m$, be a collection of m continuous functions. Prove that there exists $x^* \in [0,1]$ such that $\sum_{i=1}^{m} f_i(x^*) = mx^*$.